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EVERY CONNECTED SPACE HAS THE HOMOLOGY OF A $K(\pi, 1)$†

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§1. STATEMENT OF RESULTS

IN RECENT years Barratt–Kahn–Priddy[8], Mather[6], Priddy[9], Quillen[11], Thurston[12], Wagoner[13] and others discovered that various *infinite loop spaces* and *spaces of homeomorphisms* have the *homology of a* $K(\pi, 1)$ and those results make one wonder whether this is also the case for other spaces.

The main purpose of this note is to give a positive answer to this question and to show that in fact *every path connected space has the homology of a* $K(\pi, 1)$. More precisely:

1.1 THEOREM. *For every path connected space X with base point there exists a (Serre) fibration*

$$TX \xrightarrow{tX} X$$

which is natural with respect to X and has the following properties.

(i) *the map tX induces an isomorphism on (singular) homology and cohomology with local coefficients*

$$H_*(TX; A) \approx H_*(X; A) \qquad H^*(TX; A) \approx H^*(X; A)$$

for every local coefficient system A on X, and

(ii) *$\pi_i TX$ is trivial for $i \neq 1$ and $\pi_1 tX$ is onto.*

Furthermore *the homotopy type of X is completely determined by the pair of groups* (G_X, P_X) where

$$G_X = \pi_1 TX \qquad \text{and} \qquad P_X = \ker \pi_1 tX.$$

In fact it is not hard to prove, using Theorem 1.1, the Serre spectral sequence and the commutativity of the diagram

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \pi_1 X \arrow[r] \arrow[d] & \pi_1 X \arrow[d] \\
UX \arrow[r] \arrow[d] & \tilde{T}X \arrow[r] \arrow[d] & \tilde{X} \arrow[d] \\
UX \arrow[r] & TX \arrow[r] & X
\end{tikzcd}$$

in which $\tilde{X}$ denotes the universal covering of X, $\tilde{T}X$ the induced covering of TX and UX is the fibre of tX, that P_X *is a perfect* (i.e. $H_1 = 0$) *normal subgroup of* G_X and that

1.2. *The space X can, up to homotopy, be recovered from the pair of groups (G_X, P_X)* (i) *by applying the* $(\ )^+$ *construction of Quillen* [11, 13] *to $K(G_X, 1)$ with respect to the perfect normal subgroup $P_X \subset G_X$, or equivalently* (ii) *by applying the fibre-wise Z-completion functor of Bousfield–Kan* [1] *to the fibration $K(G_X, 1) \to K(G_X/P_X, 1)$.*

1.3. *The simply connected space $\tilde{X}$ can, up to homotopy, be obtained by applying the Z-completion functor* [1] *to $K(P_X, 1)$.*

1.4. *The acyclic space UX can, up to homotopy, be obtained by applying the acyclic functor of Dror* [3] *to $K(P_X, 1)$.*

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The proof of Theorem 1.1 is rather technical and we therefore will in §2 discuss the ideas behind it and indicate how the actual proof is spread out over the remaining sections.

§2. COMMENTS ON THE PROOF OF THEOREM 1.1

2.1. The ideas behind the proof

We can probably best illustrate these by describing how, for every integer $n \geq 1$, one can obtain a group σ^n such that $K(\sigma^n, 1)$ has the homology of the n-sphere S^n.

Clearly one can take $\sigma^1 = Z$, the integers. Now suppose that we already have σ^{n-1}. Copying the construction of S^n out of S^{n-1} by attaching two cones, we (see 3.3) embed σ^{n-1} in a kind of *homological cone*, i.e. a group $C'\sigma^{n-1}$ such that $K(C'\sigma^{n-1}, 1)$ is acyclic and then glue two copies of $K(C'\sigma^{n-1}, 1)$ together along $K(\sigma^{n-1}, 1)$. The resulting space obviously has the homology of S^n and, by an *asphericity* result of J. H. C. Whitehead (4.1), its higher homotopy groups vanish and its fundamental group is thus the desired group, the free product with amalgamation

$$\sigma^n = C'\sigma^{n-1} \amalg_{\sigma^{n-1}} C'\sigma^{n-1}.$$

2.2. Further comments

It is not hard to see that one can apply essentially the same methods to any CW-complex X, which has only one vertex and for which the attaching map $S^{n-1} \to X$ of any n-cell maps S^{n-1} homeomorphically onto a subcomplex of X. However, if one tries to deal with more general CW-complexes, things become rather complicated as one has to keep track of lots of homotopies.

To get around this we work (*semi-*) *simplicially*. This has the advantage that we now can do everything *functorially* and that, in view of the well known equivalence of simplicial and topological homotopy theories, our results automatically apply to topological spaces as well. But a possible disadvantage is that the groups G_X become very large, because the singular complex of a space is so huge; for instance G_{S^n} is much larger than the group σ^n constructed above.

Although the group G_X we obtain in the proof of Theorem 1.1 is *uncountable* even for a finite CW-complex X, our results imply (see 5.1) that *for every countable CW-complex X there is a countable group G'_X such that* $K(G'_X, 1)^+ \simeq X$ where the $(\ )^+$ construction is taken with respect to a suitable perfect normal subgroup. Two other interesting questions along this line of inquiry are:

(i) If X is a *finite CW-complex*, is there a finite CW-complex $T'X$, homologically equivalent to X, where $T'X$ is a $K(\pi, 1)$?

(ii) If X is a *compact n-manifold* $\neq S^2$ or P^2, is there a compact n-manifold $T''X$, homologically equivalent to X, with $T''X$ a $K(\pi, 1)$?

If X is a finite CW-complex and $\dim X \leq 2$, then (i) has a positive answer and (ii) has a positive answer, if X is a manifold of dimension ≤ 3.

2.3. Organization of the proof

§3 deals with *homological cones* for groups as well as for simplicial groups and §4 contains a simplicial group version of J. H. C. Whiteheads *asphericity* result. Both of these are then used in §5 to obtain a simplicial version of Theorem 1.1, from which we finally derive Theorem 1.1 itself.

For details on *simplicial homotopy theory* and its relationship to topological homotopy theory we refer the reader to [7] and [1, Ch. VIII].

§3. HOMOLOGICAL CONES FOR GROUPS AND SIMPLICIAL GROUPS

In this section we show how one can *embed every group and every simplicial group in an acyclic one*, i.e. one which has an acyclic classifying space. We start with constructing

3.1. A homological cone functor for groups

Let Q denote the rationals, let Aut Q be the group of the (set) automorphisms of Q which have compact support, i.e. which are the identity outside some finite interval, and, for every group G, let G^Q denote the group of the functions $Q \to G$ which have compact support, i.e. which go to 1 outside some finite interval. Then Aut Q acts on G^Q by composition and we define CG as the resulting semi-direct product (or split extension), i.e. the elements of CG are the pairs (b, a)

where $b \in G^Q$ and $a \in \operatorname{Aut} Q$ and the multiplication is given by the formula

$$(b, a)(b', a') = (ba(b'), aa') \qquad b, b' \; G^Q, a, a' \in \operatorname{Aut} Q.$$

Clearly this construction is *functorial* and it is easy to obtain a *natural embedding* $G \to CG$ by assigning, for instance, to every element $g \in G$ the element $(b_g, \mathrm{id}) \in CG$, where $b_g 0 = g$ and $b_r r = 1$ for $r \neq 0$.

Finally, applying the argument of Mather[6] to Aut Q, one gets that the group Aut Q is acyclic, i.e. $H_i(\bar{W}\operatorname{Aut} Q); Z) = 0$ for $i > 0$, where $\bar{W}$ denotes the *simplicial classifying functor* of Eilenberg–MacLane[7, p. 87], and a slight refinement of this argument readily yields:

3.2. Proposition. *The group CG is acyclic, i.e. $H_i(\bar{W}CG; Z) = 0$ for $i > 0$.*

The group CG is always *uncountable*, because Aut Q is so. However one has

3.3. Proposition. *There is a (not necessarily natural) subgroup $C'G \subset CG$, which contains G, is acyclic and has the same cardinality as G (except if G is finite, in which case $C'G$ may have to be countable).*

Proof. This follows readily, by a direct limit argument, from 3.2 and the fact that for every subgroup $B \subset CG$ and *cycle* z on B there is a *finite* set of elements of CG which together with B generate a subgroup of CG on which z *bounds.*

Applying the functor C *dimension-wise* to a simplicial group Y one gets a simplicial group CY for which the following generalization of 3.2 holds

3.4. Proposition. *The simplicial group CY is acyclic, i.e. $H_i(\bar{W}CY;\ Z) = 0$ for $i > 0$.*

Furthermore the argument of 3.3 yields

3.5. Proposition. *There is a (not necessarily natural) simplicial subgroup $C'Y \subset CY$, which contains Y, is acyclic and has the same cardinality as Y.*

Proof of 3.4. By applying the classifying functor $\bar{W}$ to each of the groups CY_n one gets a bi-simplicial set. The homology spectral sequence of this bi-simplicial set converges strongly to the homology of the diagonal D [2, 2.9] and 3.2 thus implies that $H_i(D; Z) = 0$ for $i > 0$. It remains to show that D has the same homotopy type as $\bar{W}CY$. For this one applies the functor W of [7, p. 88] to each of the groups CY_n and denotes by E the diagonal of the resulting bi-simplicial group. As each WCY_n is contractible [7, 21.5], so is E [10]. Moreover, it is not hard to verify that the obvious map $E \to D$ is a principal fibration with CY as fibre and thus D has the same homotopy type as $\bar{W}CY$.

A convenient property of the functor C on simplicial groups is that it preserves homotopy in the following strong sense:

3.6. Homotopy lemma. *Let $f: Y \to Y'$ be a homomorphism between simplicial groups which induces an isomorphism on the homotopy groups in all dimensions ≥ 0. Then so does the map $Cf: CY \to CY'$.*

A useful consequence for our purposes is

3.7. Corollary. *Let Y be an 'aspherical' simplicial group, i.e. $\pi_i Y = 0$ for $i > 0$. Then so is CY and the projection $Y \to \pi_0 Y$ induces an isomorphism $\pi_0 CY \approx C\pi_0 Y$. Furthermore, if $C'Y$ is as in* 3.5, *then the induced map $\pi_0 Y \to \pi_0 C'Y$ is a monomorphism.*

Proof of 3.6. This lemma follows readily from the observation that, for a group G, the underlying pointed set of CG depends only on the underlying pointed set of G and the fact that (i) a homomorphism between simplicial groups induces an isomorphism on all homotopy groups if and only if it induces a homotopy equivalence between the underlying (pointed) simplicial sets [7, 12.5 and 17.1], and (ii) every dimension-wise functor on (pointed) simplicial sets preserves homotopies [4, 5.3].

3.8. *Remark.* The homological cone functor constructed above is *not* the only one possible. Variants can be obtained by, for instance, considering only continuous automorphisms of Q or replacing Q by a finite dimensional vector space over Q or over the reals. One can also embed a group in its group ring and then take a cone on the latter as in [13]. And, of course, a composition of two cone functors is again a cone functor.

§4. J. H. C. WHITEHEAD'S ASPHERICITY RESULT

In this section we formulate and prove a simplicial group version (4.2) of an asphericity result of J. H. C. Whitehead. We first recall the original version [14, Th. 5].

4.1. Proposition. *Let*

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ A'' & \longrightarrow & X \end{array}$$

be a push out diagram of reduced (only one vertex) simplicial sets, such that (i) *A, A' and A'' are aspherical, i.e. their homotopy groups vanish in dimensions* >1, *and* (ii) *the maps $A \to A'$, $\pi_1 A \to \pi_1 A'$, $A \to A''$, $\pi_1 A \to \pi_1 A''$, are all monomorphisms.*

Then X is also aspherical and the induced diagram

$$\begin{array}{ccc} \pi_1 A & \longrightarrow & \pi_1 A' \\ \downarrow & & \downarrow \\ \pi_1 A'' & \longrightarrow & \pi_1 X \end{array}$$

is a push out diagram of groups, i.e. $\pi_1 X$ is the free product with amalgamation $\pi_1 X = \pi_1 A' \amalg_{\pi_1 A} \pi_1 A''$.

The simplicial group version of this result is

4.2. Proposition. *Let*

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \downarrow & & \downarrow \\ B'' & \longrightarrow & Y \end{array}$$

be a push out diagram of simplicial groups such that (i) *B, B' and B'' are 'aspherical' in the sense that their homotopy groups vanish in dimensions* ≥ 1, *and* (ii) *the maps $B \to B'$, $\pi_0 B \to \pi_0 B'$, $B \to B''$, $\pi_0 B \to \pi_0 B''$, are all monomorphisms.*

Then Y is also 'aspherical' and the induced diagram

$$\begin{array}{ccc} \pi_0 B & \longrightarrow & \pi_0 B' \\ \downarrow & & \downarrow \\ \pi_0 B'' & \longrightarrow & \pi_0 Y \end{array}$$

is a push out diagram of groups, i.e. $\pi_0 Y$ is the free product with amalgamation

$$\pi_0 Y = \pi_0 B' \amalg_{\pi_0 B} \pi_0 B''.$$

This follows immediately from 4.1 and the following proposition, which seems to be of interest in itself.

4.3. Proposition. *Let*

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \downarrow & & \downarrow \\ B'' & \longrightarrow & Y \end{array}$$

be a push out diagram of simplicial groups such that the maps $B \to B'$ and $B \to B''$ are both monomorphisms. Then the induced map $\bar{W}B' \amalg_{\bar{W}B} \bar{W}B'' \to \bar{W}Y$ is a weak homotopy equivalence.

Here again $\bar{W}$ denotes the simplicial classifying functor of Eilenberg–MacLane [7, p. 87].

Proof. By applying $\bar{W}$ dimension-wise to the simplicial groups B, B', B'' and Y, one gets bi-simplicial sets of which the diagonals D, D', D'' and E have the homotopy type of $\bar{W}B$, $\bar{W}B'$, $\bar{W}B''$ and $\bar{W}Y$ respectively (see the proof of 3.4) and it thus suffices to prove that the induced map $D' \amalg_D D'' \to E$ is a weak homotopy equivalence. But this is not hard to do, using [1, Ch. XII 4.2 and 4.3] and the fact that, in view of 4.2, the induced map $\bar{W}B'_n \amalg_{\bar{W}B_n} \bar{W}B''_n \to \bar{W}Y$ is a weak homotopy equivalence for all $n \geq 0$.

§5. PROOF OF THEOREM 1.1

We prove Theorem 1.1 by first obtaining an analogous result (5.1) for reduced (only one vertex) simplicial sets and then translating this into topological language with the help of the *geometric realization* $\| \;$ and the *singular functor* Sin.

5.1. THEOREM. *For every reduced simplicial set X there exists a fibration*

$$TX \xrightarrow{tX} X$$

which is natural with respect to X and has the following properties: (i) *the map tX induces an isomorphism on homology and cohomology with local coefficients $H_*(TX; A) \approx H_*(X; A)$, $H^*(TX; A) \approx H^*(X; A)$ for every local coefficient system A on X,* (ii) *$\pi_i TX$ is trivial for $i \neq 1$ and $\pi_1 tX$ is onto, and* (iii) *$\pi_1 TX$ has the same cardinality as (the set of simplicities of) X.*

Proof. We recall from [7, p. 87 and p. 118] that the *loop group functor G* (which assigns to every reduced simplicial set X a free simplicial group which has the homotopy type of the loops on X) is the *left adjoint* of the *simplicial classifying functor $\bar{W}$* of Eilenberg–MacLane and that the adjunction map $X \to \bar{W}GX$ is a weak homotopy equivalence.

The core of the proof then consists of constructing a (natural) sequence of simplicial groups and homomorphisms

$$1 = G^0X \to G^1X \to \cdots \to G^{n-1}X \to G^nX \to \cdots$$

together with a compatible set of homomorphisms $G^nX \to GX^n$ (where X^n denotes the n-skeleton of X), such that the induced map $\bar{W}G^\infty X = \bar{W} \varinjlim G^nX \to \bar{W}GX = \bar{W} \varinjlim GX^n$ has the properties (i), (ii) and (iii) of Theorem 5.1, as the desired map $tX\colon TX \to X$ then readily can be obtained from the pull back diagram:

$$\begin{array}{ccc} TX & \longrightarrow & \bar{W}G^\infty X \\ \downarrow & & \downarrow \\ X & \longrightarrow & \bar{W}GX \end{array}$$

For every integer $n > 0$ let $D[n]$ denote the reduced simplicial set obtained from the standard n-simplex $\Delta[n]$ by identifying all its vertices. The simplicial groups G^nX then are inductively defined by means of the push out diagrams

$$\text{(I)} \qquad \begin{array}{ccc} \amalg G^{n-1}D[n] & \longrightarrow & G^{n-1}X\pi \amalg C'G^{n-1}D[n] \\ \downarrow & & \downarrow \\ \amalg GD[n]\pi \amalg C'G^{n-1}D[n] & \longrightarrow & G^nX \end{array}$$

(where $C'G^{n-1}D[n]$ is as in 3.5) and the homomorphisms $G^nX \to GX^n$ are induced by the obvious maps from these diagrams to the commutative diagrams

$$\text{(II)} \qquad \begin{array}{ccc} \amalg GD[n]^{n-1} & \longrightarrow & GX^{n-1} \\ \downarrow & & \downarrow \\ \amalg GD[n] & \longrightarrow & GX^n \end{array}$$

In all these diagrams the free products $\amalg$ are taken over all maps $D[n] \to X$. It now follows immediately from 3.5, 3.7 and 4.2 that the induced map $\bar{W}G^\infty X \to \bar{W}GX$ has properties (ii) and (iii) and it thus remains to verify property (i).

To do this we consider the reduced simplicial set $\bar{X}$ which has exactly one *non-degenerate* simplex $\bar{x}$ for every simplex $x \in X$ with, of course, the obvious faces and degeneracies. The map $\bar{X} \to X$ which sends $\bar{x}$ to x then is a weak homotopy equivalence [5] and one clearly has push out diagrams

$$\text{(III)} \qquad \begin{array}{ccc} \amalg GD[n]^{n-1} & \longrightarrow & G\bar{X}^{n-1} \\ \downarrow & & \downarrow \\ \amalg GD[n] & \longrightarrow & G\bar{X}^{n} \end{array}$$

where again the free products $\amalg$ are taken over all maps $D[n] \to X$. Moreover the maps from the diagrams (I) to the diagrams (II) clearly factor through the diagrams (III) and it is not hard to show, using 3.5, 4.3 and a Mayer–Vietoris argument, that the resulting map $G^nX \to G\bar{X}^n$ induces isomorphisms $H_*(\bar{W}G^nX; A) \approx H_*(\bar{W}G\bar{X}^n; A)$, $H^*(\bar{W}G^nX; A) \approx H^*(\bar{W}G\bar{X}^n; A)$ for every local coefficient system A which comes from X. The desired result now follows immediately.

We end with completing the

Proof of theorem 1.1. Let $E_0 \operatorname{Sin} X$ denote the largest subcomplex of $\operatorname{Sin} X$ (the singular complex of X) which has no other vertex than the base point, and consider the composition

$$|TE_0 \operatorname{Sin} X| \xrightarrow{|tE_0 \operatorname{Sin} X|} |E_0 \operatorname{Sin} X| \xrightarrow{\text{incl.}} |\operatorname{Sin} X| \to X$$

where T and t are as in Theorem 5.1 and the map $|\operatorname{Sin} X| \to X$ is the adjunction map. Then it is not hard to see that any natural factorization of this map

$$|TE_0 \operatorname{Sin} X| \to TX \xrightarrow{tX} X$$

for which the first map is a weak homotopy equivalence and the second map is a Serre fibration, yields the desired result.

REFERENCES

1. A. K. Bousfield and D. M. Kan: Homotopy limits, completions and localizations, *Lecture Notes in Math.* **304**, Springer (1972).
2. A. Dold and D. Puppe: Homologie nicht-additive Functoren. Anwendungen, *Ann. Inst. Fourier* **11** (1961), 201–312.
3. E. Dror: Acyclic spaces, *Topology* **11** (1972), 339–348.
4. D. M. Kan: On the homotopy relation for c.s.s. maps, *Bull. Soc. Mat. Mex.* (1957), 75–81.
5. Y. Kodawa: A relation between two realizations of complete semisimplicial complexes, *Proc. Japan Ac.* **33** (1957), 536–540.
6. J. Mather: The vanishing of the homology of certain groups of homeomorphisms, *Topology* **10** (1971), 297–298.
7. J. P. May: Simplicial objects in algebraic topology, Van Nostrand (1967).
8. S. B. Priddy: On $\Omega^\infty S^\infty$ and the infinite symmetric group, *Proc. Symp. Pure Math. AMS* **22** (1971), 217–220.
9. S. B. Priddy: Fredholm permutations and stable homotopy, *Mich. Math. J.* **20** (1973), 187–192.
10. D. G. Quillen: Spectral sequences of a double semi-simplicial group, *Topology* **5** (1966), 155–157.
11. D. G. Quillen: Cohomology of groups, *Proc. Int. Congress of Math.* (1970).
12. W. Thurston: Foliations and groups of diffeomorphisms, *Bull. AMS* **80** (1974), 304–307.
13. J. B. Wagoner: Delooping classifying spaces in algebraic K-theory, *Topology* **11** (1972), 349–370.
14. J. H. C. Whitehead: On the asphericity of regions in a 3-sphere, *Fund. Math.* **32** (1939), 149–166.

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